38. The average time T, in milliseconds (ms), it takes a person to move a mouse cursor a distance D across a computer screen to a target button of length S is given by *Fitts' Law*, which states that $T = a + b\log(D/S + 1)$, where a and b are constants.[6]

(a) Briefly explain why it does not matter what units are used for S and D, provided the units are the same. Why is it useful for a formula concerning computer screens not to be based on fixed sizes?

(b) The cursor is moved 15 cm to a target of length 3 cm. Letting $a = 50$ and $b = 500$, estimate the time required to complete this operation.

(c) Graph T for $S = 3$ cm and for $0 \le D \le 20$.

(d) What is the T-intercept of the graph in part (c)? What does this tell you about cursor movement?

(e) Suppose the distance moved, D, is doubled and the target length S stays the same. Without knowing the values of D and S, but assuming $a = 50$ and $b = 500$, what can you say about the change in T, the time required?

4.4 LOGARITHMIC SCALES

The Solar System and Beyond

Table 4.4 gives the distance from the sun to a number of different astronomical objects. The planet Mercury is 58,000,000 km from the sun, that earth is 149,000,000 km from the sun, and that Pluto is 5,900,000,000 km, or almost 6 billion kilometers from the sun. The table also gives the distance to Proxima Centauri, the star closest to the sun, and to the Andromeda Galaxy, the spiral galaxy closest to our own galaxy, the Milky Way.

Table 4.4 *Distance from the sun to various astronomical objects*

Object	Distance (million km)	Saturn	1426
Mercury	58	Uranus	2869
Venus	108	Neptune	4495
Earth	149	Pluto	5900
Mars	228	Proxima Centauri	$4.1 \cdot 10^7$
Jupiter	778	Andromeda Galaxy	$2.4 \cdot 10^{13}$

Linear Scales

We can represent the information in Table 4.4 graphically in order to get a better feel for the distances involved. Figure 4.13 shows the distance from the sun to the first five planets on a *linear scale*, which means that the evenly spaced units shown in the figure represent equal distances. In this case, each unit represents 100 million kilometers.

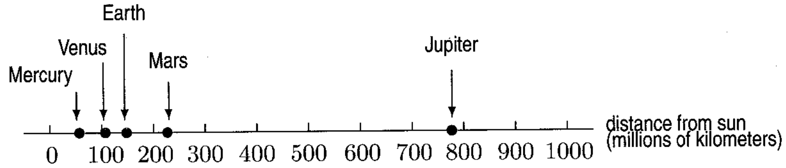

Figure 4.13: The distance from the sun of the first five planets (in millions of kilometers)

[6] Raskin, J, *The Humane Interface* (Reading: Addison Wesley, 2000), p 93. Jef Raskin designed the Apple Macintosh computer using, among other resources, mathematical formulas such as Fitts' Law.

The drawback of Figure 4.13 is that the scale is too small to show all of the astronomical distances described by the table. For example, to show the distance to Pluto on this scale would require over six times as much space on the page. Even worse, assuming that each 100 million km unit on the scale measures half an inch on the printed page, we would need 3 miles of paper to show the distance to Proxima Centauri!

You might conclude that we could fix this problem by choosing a larger scale. In Figure 4.14 each unit on the scale is 1 billion kilometers. Notice that all five planets shown by Figure 4.13 are crowded into the first unit of Figure 4.14; even so, the distance to Pluto barely fits. The distances to the other objects certainly don't fit. For instance, to show the Andromeda Galaxy, Figure 4.14 would have to be almost 200,000 miles long. Choosing an even larger scale will not improve the situation.

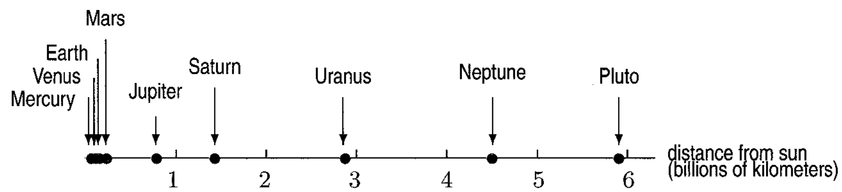

Figure 4.14: The distance to all nine planets (in billions of kilometers)

Logarithmic Scales

We conclude that the data in Table 4.4 cannot easily be represented on a linear scale. If the scale is too small, the more distant objects do not fit; if the scale is too large, the less distant objects are indistinguishable. The problem is not that the numbers are too big or too small; the problem is that the numbers vary too greatly in size.

We consider a different type of scale on which equal distances are not evenly spaced. All the objects from Table 4.4 are represented in Figure 4.15. The nine planets are still cramped, but it is possible to tell them apart. Each tick mark on the scale in Figure 4.15 represents a distance ten times larger than the one before it. This kind of scale is called *logarithmic*.

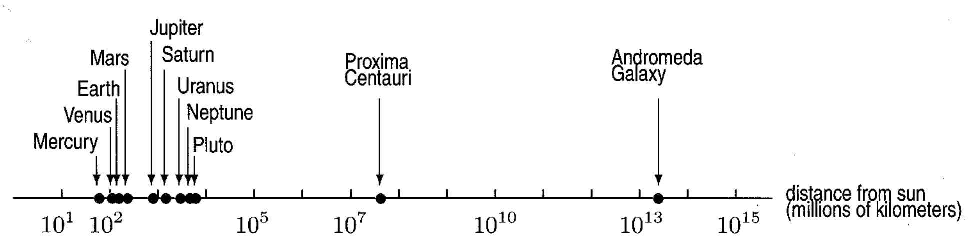

Figure 4.15: The distance from the sun (in millions of kilometers)

How Do We Plot Data on a Logarithmic Scale?

A logarithmic scale is marked with increasing powers of 10: 10^1, 10^2, 10^3, and so on. Notice that even though the distances in Figure 4.15 are not evenly spaced, the exponents are evenly spaced. Therefore the distances in Figure 4.15 are spaced according to their logarithms.

In order to plot the distance to Mercury, 58 million kilometers, we use the fact that

$$10 < 58 < 100,$$

so Mercury's distance is between 10^1 and 10^2, as shown in Figure 4.15. To plot Mercury's distance more precisely, calculate $\log 58 = 1.763$, so $10^{1.763} = 58$, and use 1.763 to represent Mercury's position. See Figure 4.16.

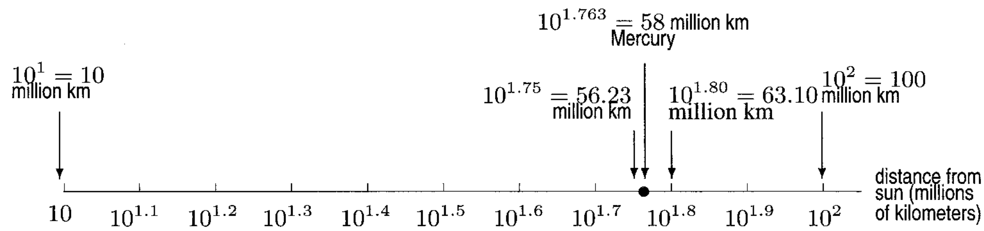

Figure 4.16: Mercury's distance, 58 million kilometers falls between $10^{1.75} = 56.23$ and $10^{1.80} = 63.10$ million kilometers

Example 1 Where should Saturn be on the logarithmic scale? What about the Andromeda Galaxy?

Solution Saturn's distance is 1426 million kilometers, so we want the exponent of 10 that gives 1426, which is

$$\log 1426 \approx 3.154119526.$$

Thus $10^{3.154} \approx 1426$, so we use 3.154 to indicate Saturn's distance.

Similarly, the distance to the Andromeda Galaxy is $2.4 \cdot 10^{13}$ million kilometers, and since

$$\log(2.4 \cdot 10^{13}) \approx 13.38,$$

we use 13.38 to represent the galaxy's distance. See Figure 4.17.

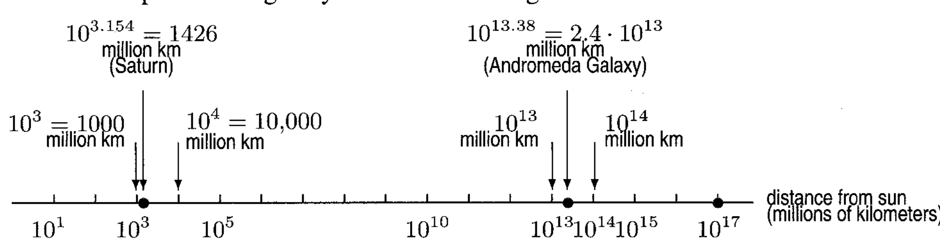

Figure 4.17: Saturn's distance is $10^{3.154}$ and the Andromeda Galaxy's distance is $10^{13.38}$

Logs of Small Numbers

The history of the world, like the distance to the stars and planets, involves numbers of vastly different sizes. Table 4.5 gives the ages of certain events[7] and the logarithms of their ages. The logarithms have been used to plot the events in Figure 4.18.

Table 4.5 *Ages of various events in earth's history and logarithms of the ages*

Event	Age (millions of years)	log (age)	Event	Age (millions of years)	log (age)
Man emerges	1	0	Rise of dinosaurs	245	2.39
Ape man fossils	5	0.70	Vertebrates appear	570	2.76
Rise of cats, dogs, pigs	37	1.57	First plants	2500	3.40
Demise of dinosaurs	67	1.83	Earth forms	4450	3.65

[7] *CRC Handbook*, 75^{th} *ed.* 14-8.

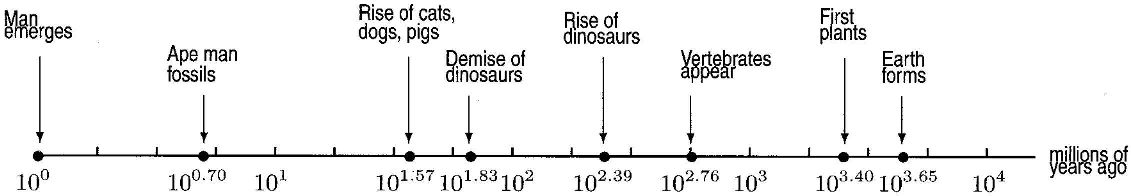

Figure 4.18: Logarithmic scale showing the ages of various events (in millions of years ago)

The events described by Table 4.5 all happened at least 1 million years ago. How do we indicate events which occurred less than 1 million years ago on the log scale?

Example 2 Where should the building of the pyramids be indicated on the log scale?

Solution The pyramids were built about 5000 years ago, or

$$\frac{5000}{1{,}000{,}000} = 0.005 \text{ million years ago.}$$

Notice that 0.005 is between 0.001 and 0.01, that is,

$$10^{-3} < 0.005 < 10^{-2}.$$

Since

$$\log 0.005 \approx -2.30,$$

we use -2.30 for the pyramids. See Figure 4.19.

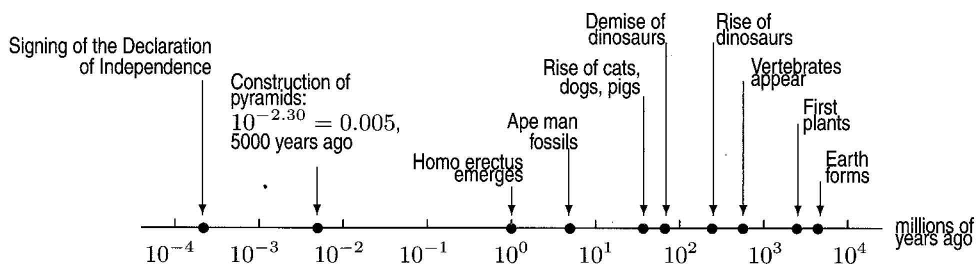

Figure 4.19: Logarithmic scale showing the ages of various events. Note that events that are less than 1 million years old are indicated by negative exponents

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Another Way to Label a Log Scale

In Figures 4.18 and 4.19, the log scale has been labeled so that exponents are evenly spaced. Another way to label a log scale is with the values themselves instead of the exponents. This has been done in Figure 4.20.

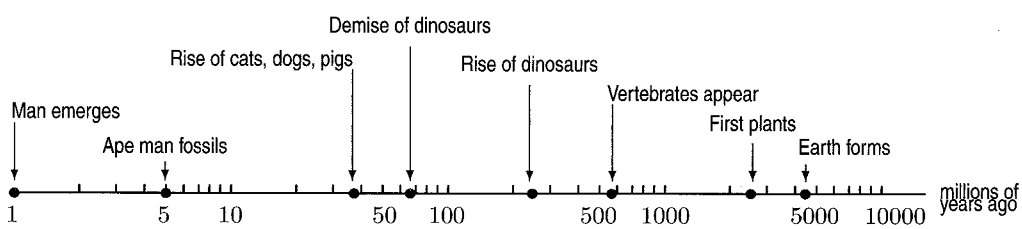

Figure 4.20: Axis labeled using the actual values, not with logs

Notice the characteristic way that the labels and tick marks "pile up" on each interval. The even spacing between exponents on log scales leads to uneven spacing in values. Although the values 10, 20, 30, 40, and 50 are evenly spaced, their corresponding exponents are not: $\log 10 = 1$, $\log 20 = 1.30$, $\log 30 = 1.48$, $\log 40 = 1.60$, and $\log 50 = 1.70$. Therefore, when we label an axis according to values on a scale that is spaced according to exponents, the labels get bunched up.

Log-Log Scales

Table 4.6 shows the average metabolic rate in kilocalories per day (kcal/day) for animals of different weights.[8] (A kilocalorie is the same as a standard nutritional calorie.) For instance, a 1-lb rat consumes about 35 kcal/day, whereas a 1750-lb horse consumes almost 9500 kcal/day.

Table 4.6 *The metabolic rate (in kcal/day) for animals of different weights*

Animal	Weight (lbs)	Rate (kcal/day)
Rat	1	35
Cat	8	166
Human	150	2000
Horse	1750	9470

It is not practical to plot these data on an ordinary set of axes. The values span too broad a range. However, we can plot the data using log scales for both the horizontal (weight) axis and the vertical (rate) axes. See Figure 4.21. Figure 4.22 shows a close-up view of the data point for cats to make it easier to see how the labels work. Once again, notice the characteristic piling up of labels and gridlines. This happens for the same reason that it happened in Figure 4.20.

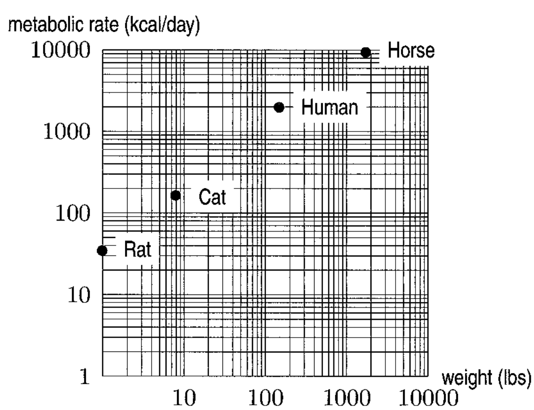

Figure 4.21: Metabolic rate (in kcal/hr) plotted against body weight

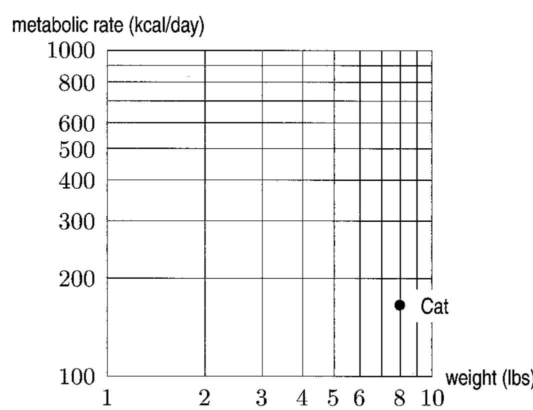

Figure 4.22: A close-up view of the Cat data point

Using Logs to Fit an Exponential Function to Data

In Section 1.6 we used linear regression to find the equation for a line of best fit for a set of data. What if the data do not lie close to a straight line, but instead approximate the graph of some other function? In this section we see how logarithms help us fit data with an exponential function of the form $Q = a \cdot b^t$.

[8] *The New York Times*, January 11, 1999.

Sales of Compact Discs

Table 4.7 shows the fall in the sales of vinyl long-playing records (LPs) and the rise of compact discs (CDs) during for the years 1982 through 1993.[9]

Table 4.7 *CD and LP sales*

t, years since 1982	c, CDs (millions)	l, LPs (millions)
0	0	244
1	0.8	210
2	5.8	205
3	23	167
4	53	125
5	102	107
6	150	72
7	207	35
8	287	12
9	333	4.8
10	408	2.3
11	495	1.2

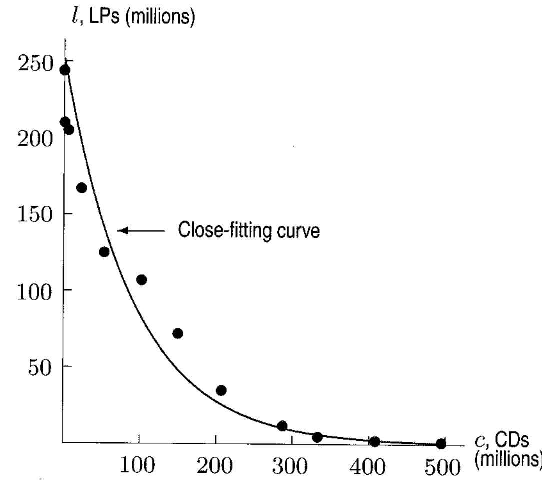

Figure 4.23: The number of LPs sold, l, as a function of number of CDs sold, c

From Table 4.7, we see that as CD sales rose dramatically during the 1980s and early 1990s, LP sales declined equally dramatically. Figure 4.23 shows the number of LPs sold in a given year as a function of the number of CDs sold that year.

Using a Log Scale to Linearize Data

In Section 4.4, we saw that a log scale allows us to compare values that vary over a wide range. Let's see what happens when we use a log scale to plot the data shown in Figure 4.23. Table 4.8 shows values $\log l$, where l is LP sales. These are plotted against c, CD sales, in Figure 4.24. Notice that plotting the data in this way tends to *linearize* the graph—that is, make it look more like a line. A line has been drawn in to emphasize the trend in the data.

Table 4.8 *Values of $y = \ln l$ and c.*

c, CDs	l, LPs	$y = \ln l$
0	244	5.50
0.8	210	5.35
5.8	205	5.32
23	167	5.12
53	125	4.83
102	107	4.67
150	72	4.28
207	35	3.56
287	12	2.48
333	4.8	1.57
408	2.3	0.83
495	1.2	0.18

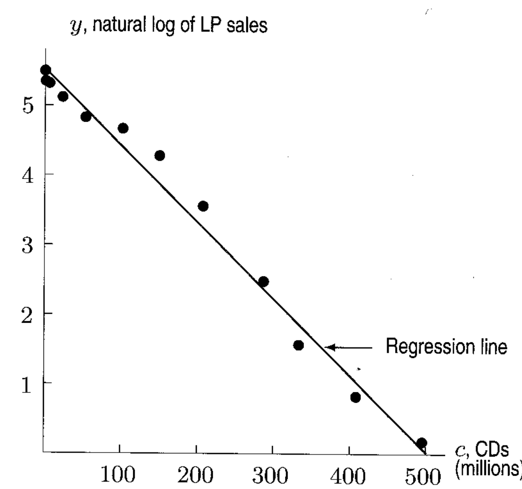

Figure 4.24: The y-axis of this graph gives the natural log of LP sales

[9]Data from Recording Industry Association of America, Inc., 1998

Finding a Formula for the Curve

We say that the data in the third column of Table 4.8 have been *transformed*. A calculator or computer gives a regression line for the transformed data:[10]

$$y = 5.52 - 0.011c.$$

Notice that this equation gives y in terms of c. To transform the equation back to our original variables, l and c, we substitute $\ln l$ for y, giving

$$\ln l = 5.52 - 0.011c.$$

We solve for l by raising e to both sides:

$$\begin{aligned} e^{\ln l} &= e^{5.52-0.011c} \\ &= (e^{5.52})(e^{-0.011c}) \qquad \text{Using an exponent rule.} \end{aligned}$$

Since $e^{\ln l} = l$ and $e^{5.52} \approx 250$, we have

$$l = 250e^{-0.011c}.$$

This is the equation of the curve in Figure 4.23.

Fitting An Exponential Function To Data

In general, to fit an exponential formula, $N = ae^{kt}$, to a set of data of the form (t, N), we use three steps. First, we transform the data by taking the natural log of both sides and making the substitution $y = \ln N$. This leads to the equation

$$\begin{aligned} y = \ln N &= \ln\left(ae^{kt}\right) \\ &= \ln a + \ln e^{kt} \\ &= \ln a + kt. \end{aligned}$$

Setting $b = \ln a$ gives a linear equation with k as the slope and b as the y-intercept.

$$y = b + kt.$$

Secondly we can now use linear regression on the variables t and y. (Remember that $y = \ln N$.) Finally, as step three, we transform the linear regression equation back into our original variables by substituting $\ln N$ for y and solving for N.

Exercises and Problems for Section 4.4

Exercises

In Exercises 1–4, you wish to graph the quantities on a standard piece of paper. On which should you use a logarithmic scale? On which a linear scale? Why?

1. The wealth of 20 different people, one of whom is a multi-billionaire.
2. The number of diamonds owned by 20 people, one of whom is a multi-billionaire.
3. The number of meals per week eaten in restaurants for a random sample of 20 people worldwide.
4. The number of tuberculosis bacteria in 20 different people, some never exposed to the disease, some slightly exposed, some with mild cases, and some dying of it.

[10]The values obtained by a computer or another calculator may vary slightly from the ones given.